$$\beta_{1,2} \approx [-p_0(\operatorname{Re} E_1\varepsilon - \alpha) \pm \Delta^{1/2}]/g_{002} \tag{3.5}$$
$$g_{002} = \lambda^2 (2\pi)^{-1}(p_0^2 - |p_k|^2)$$
$$\Delta = (\operatorname{Im} E_1)^2 (|p_k|^2 - p_0^2)\varepsilon^2 + |p_k|^2 (\alpha - \varepsilon \operatorname{Re} E_1)^2$$
$$p_k = \langle \Phi e^{-ik\omega\tau} \rangle, \quad \Phi = \sum_{j=-\infty}^{\infty} p_j e^{ij\omega\tau}$$

$\Phi(\tau)$ is the periodic load with period $p = 2\pi/\omega$, acting upon the shell, and E_1 is a coefficient depending on the inherent properties of the shell.

We have the following possibilities:

1°. If $p_0^2 < |p_k|^2$ $(g_{002} < 0)$, then the branching equation always has a solution (3.5) and we have instability near the k-th resonance $(k = 1, 2, \ldots)$ for small $\alpha, \varepsilon, \beta$.

2°. If $p_0^2 > |p_k|^2$ $(g_{002} > 0)$, then, provided that $\Delta < 0$, the branching equation has no solutions and we have stability near the k-th resonance $(k = 1, 2, \ldots)$ for small $\alpha, \varepsilon, \beta$. When $\Delta \geqslant 0$, the equation of the neutral curve will have the form (3.5).

3°. If $p_0^2 = |p_k|^2$ $(g_{002} = 0)$, and the coefficient accompanying the third power of the parameter β is not zero, then the equation of the neutral curve is obtained from a cubic equation.

Let us consider, as an example, the function $\Phi = 1 + \mu \cos \omega\tau$, in which case $p_0 = 1$, $p_1 = p_{-1} = \mu/2$, $p_k = 0$, $k = 2, 3, \ldots$. Near the principal resonance $(k = 1)$ when $|\mu| < 2$ and $\Delta < 0$ (case 2°) we have stability for small $\alpha, \varepsilon, \beta$, if on the other hand $|\mu| < 2$ and $\Delta \geqslant 0$, then the equation of the neutral curve has the form (3.5).

In the case of higher-order resonances $(k = 2, 3, \ldots)$ we always have $\Delta < 0$, and hence instability. In the case of an elastic shell we have in the same situation $(p_0 = 1, p_k = 0, k = 2, 3, \ldots)$, the condition $\Delta = 0$ holds and the neutral curve is given by the equation

$$\beta_{1,2} \approx 2\pi\alpha/\lambda^2, \quad k = 2, 3, \ldots$$

The author thanks V.I. Yudovich for his interest and for useful discussions.

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Translated by L.K.

PMM U.S.S.R.,Vol.52,No.3,pp.408-412,1988
Printed in Great Britain

0021-8928/88 $10.00+0.00

A METHOD OF ANALYSING PLATES AND SHALLOW SHELLS*

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A method of representing a function of two variables defined in a square $\sigma = [0, \pi] \times [0, \pi]$, in the form of a combination of polynomials and differentiable trigonometric series is given. Unlike the representations obtained earlier /1-3/, the present paper proposes the use of expansions in trigonometric series over the system of functions $\{\sin mx\}$, $\{1, \cos mx\}$, $m = 1, 2, \ldots$ complete in $[0, \pi]$, and in double series over the system of functions $\{\sin mx \sin ny\}$, $\{\sin ny, \cos mx \sin ny\}$, $\{\sin mx, \sin mx \cos ny\}$, $m, n = 1, 2, \ldots$ complete in σ. Expansion in such systems of functions has certain advantages compared with expansions in the usual trigonometric system of sines and cosines in $[-\pi, \pi]$ and the corresponding system of functions in the square $[-\pi, \pi] \times [-\pi, \pi]$. The proposed method is used to solve problems of the theory of shells with constant coefficients in the case of rigid clamping along a rectangular contour. The solution is obtained in the form of trigonometric series whose coefficients are expressed in terms of the solution of an infinite linear algebraic system of equations. Numerical values of the deflection are obtained for the case of a shallow circular cylindrical shell.

**Prikl.Matem.Mekhan.*,52,3,523-527,1988

1. Formulation and justification of the method. We shall use the concept of evenness of the functions $f(x)$ on $[0, \pi]$ and $F(x, y)$ on σ, and the concept of strictly defined evenness of the functions introduced in /3/. Let us formulate a lemma on the possiblity of termwise differentiation of the Fourier series of the function $f(x)$ over the systems of functions $\{\sin mx\}$ and $\{1, \cos mx\}$, $m = 1, 2, \ldots$ complete in $[0, \pi]$. In what follows, we shall assume that the index l takes the values $1, 2, \ldots, p$. The indices k and c take the values $0, 1, \ldots, p$ and summation over them is carried out from 0 to p.

Lemma. Let $f(x)$ be a function of strictly defined evenness. Let continuous derivatives f_x^{2l} (f_x^{2l-1}) exist on $[0, \pi]$ and let $f(0) = 0, f_x^{2l}(0) = 0$ $(f_x^{2l-1}(0) = 0)$. Let us also assume that the derivative f_x^{2p+2} (f_x^{2p+1}) can be represented in the form of a sine (cosine) Fourier series. Then the Fourier sine (cosine) series of the function $f(x)$ can be differentiated term by term $2p + 2$ $(2p + 1)$ times.

The proof of the lemma is analogous to that of Lemma 1 in /3/.

Theorem 1. Let $F(x, y)$ be a function with a strictly defined evenness in x and let a partial derivative $F_x^{2p}(0, y)$ $(F_x^{2p-1}(0, y))$ exist. Then the following unique representation will also exist:

$$F(x, y) = \sum_k h_k(x) \psi_k(y) + H(x, y) \tag{1.1}$$

$$F(x, y) = \sum_k q_k(x) \eta_k(y) + Q(x, y)$$

where: 1) $h_k(x)$ are any fixed polynomials of the same evenness as $F(x, y)$ in x with the following properties: the even polynomial $h_k(x)$ is of degree $2k$, and the odd polynomial is of degree $2k + 1$, $d^{2k}h_k(0)/dx^{2k} \neq 0$; 2) $q_0(x) \equiv 0$, $q_k(x) = h_k'(x)$; 3) the function $H(x, y)$ $(Q(x, y))$ satisfies the condition $H(0, y) = 0$, $H_x^{2l}(0, y) = 0$ $(Q_x^{2l-1}(0, y) = 0)$.

Theorem 2. Let $F(x, y)$ be a function with strictly defined evenness in y, and let a partial derivative $F_y^{2p}(x, 0)$ $(F_y^{2p-1}(x, 0))$ exist. Then a unique representation

$$F(x, y) = \sum_c g_c(y) \varphi_c(x) + G(x, y) \tag{1.2}$$

$$\left(F(x, y) = \sum_c r_c(y) \xi_c(x) + R(x, y)\right)$$

will also exist, where 1) $g_c(y) = h_c(y)$; 2) $r_0(y) \equiv 0$, $r_c(y) = g_c'(y)$; 3) the function $G(x, y)$ $(R(x, y))$ satisfies the conditions $G(x, 0) = 0$, $G_y^{2l}(x, 0) = 0$ $(R_x^{2l-1}(0, y) = 0)$.

Let us prove Theorem 1. Theorem 2 is proved in the same manner.

We shall denote by f_x^{2l} a $2l$-th order derivative of the function $f(x)$. Let us write two linear systems of equations in unknown functions $\psi_k(y)$, $\eta_k(y)$

$$F(0, y) = \sum_k h_k(0) \psi_k(y), \quad F_x^{2l}(0, y) = \sum_k h_k^{2l}(0) \psi_k(y) \tag{1.3}$$

$$F_x^{2l-1}(0, y) = \sum_k q_k^{2l-1}(0) \eta_k(y) \tag{1.4}$$

Each system has, by virtue of 1) and 2), a non-zero determinant since the matrices of the systems are triangular. From this it follows that (1.3) yields $\psi_k(y)$ uniquely, and (1.4) yields $\eta_k(y)$. Substituting these functions into (1.1), we obtain uniquely $H(x, y)$, $Q(x, y)$. At the same time, the functions $H(x, y)$, $Q(x, y)$, obtained in this manner will satisfy conditions 3).

From the theorems and the lemma we obtain at once the following corollary.

Corollary. Let the function $F(x, y)$ with strictly defined evenness in x, have a continuous derivative F_x^{2p+2} (F_x^{2p+1}), represented by its Fourier series in the system of functions $\{\sin mx \sin ny\}$ $(\{\sin ny, \cos mx \sin ny\})$ in σ. Then the following representation exists and is unique:

$$F(x, y) = \sum_k h_k(x) \sum_n \psi_n^k \sin ny + \sum_{m, n} H_{mn} \sin mx \sin ny$$

$$\left(F(x, y) = \sum_k q_k(x) \sum_n \eta_n^k \sin ny + \sum_n Q_{0n} \sin ny + \sum_{m, n} Q_{mn} \cos mx \sin ny\right)$$

The series in this representation can be differentiated term by term in x $2p + 2$ $(2p + 1)$ times. If on the other hand $F(x, y)$ has a derivative F_y^{2p+2} (F_y^{2p+1}) which can be represented in σ by its Fourier series in the system of functions $\{\sin mx \sin ny\}$ $(\{\sin mx, \sin mx \cos ny\})$, then the following representation exists and is unique:

$$F(x, y) = \sum_c g_c(y) \sum_m \varphi_m^c \sin mx + \sum_{m, n} G_{mn} \sin mx \sin ny$$

$$\left(F(x, y) = \sum_c r_c(y) \sum_m \xi_m^c \sin mx + \sum_m R_{m0} \sin mx + \sum_{m, n} R_{mn} \sin mx \cos ny\right)$$

The series in this representation can be differentiated term by term in y $2p+2(2p+1)$ times. The indices m and n take the values $1, 3, 5, \ldots$ and $2, 4, 6, \ldots$, depending on the evenness of $F(x, y)$ in the corresponding argument.

2. The boundary value problem of the theory of shells. Let us consider the boundary value problem of the theory of shells with constant coefficients in the displacements in the square region σ with the boundary Γ

$$L_{i1}u + L_{i2}v + L_{i3}w = P^i(x, y), \quad i = 1, 2, 3 \tag{2.1}$$

$$u|_\Gamma = v|_\Gamma = w|_\Gamma = \partial w / \partial n|_\Gamma = 0 \tag{2.2}$$

where $\partial/\partial n$ is a derivative in the direction of the outer normal to the shell surface. Let us expand each load P^1, P^2, P^3 in four terms of different, strictly defined evenness. We shall seek a solution of problem (2.1), (2.2) in the form of a sum of four problems of the form (2.1) with boundary conditions

$$u(x, 0) = 0, \; v(0, y) = 0, \; w(x, 0) = w(0, y) = 0 \tag{2.3}$$

$$u(0, y) = 0, \; v(x, 0) = 0, \; w_y(x, 0) = w_x(0, y) = 0 \tag{2.4}$$

(the subscript denotes the partial derivative in the corresponding variable).

In the present case every one of the loads P^1, P^2, P^3 will represent one of the four components of the initial loads. We shall use the following expansions in series of the component loads:

$$P^1(x, y) = \Sigma P_{0n}{}^1 \sin ny + \Sigma P_{mn}{}^1 \cos mx \sin ny$$

$$P^2(x, y) = \Sigma P_{m0}{}^2 \sin mx + \Sigma P_{mn}{}^2 \sin mx \cos ny$$

$$P^3(x, y) = \Sigma P_{mn} \sin mx \sin ny$$

Here and henceforth the summation in the course of solving the problem (2.1), (2.3), (2.4) will be carried out over m and n, which take the following values:

$$m = 1, 3, 5, \ldots, \text{ if } P^3(x, y) - \text{even in } x \text{ in } [0, \pi],$$
$$m = 2, 4, 6, \ldots, \text{ if } P^3(x, y) - \text{odd in } x \text{ in } [0, \pi];$$

with the index n changing analogously depending on the evenness of $P^3(x, y)$ in y on $[0, \pi]$.

The system (2.1) contains higher-order derivatives of the displacements: u_{xxx}, u_{xyy}, v_{yyy}, v_{xxy}, w_{xxxx}, w_{yyyy}, w_{xxyy} (the operator L_{33} contains the fourth-order derivatives of w). Following the results of Sect.1, we shall seek the solution of problem (2.1), (2.3) and (2.4) in the form

$$u = q(x)\, \Sigma \alpha_n \sin ny + \Sigma \alpha_{0n} \sin ny + \Sigma \alpha_{mn} \cos mx \sin ny \tag{2.5}$$

$$v = r(y)\, \Sigma \delta_m \sin mx + \Sigma \delta_{m0} \sin mx + \Sigma \delta_{mn} \sin mx \cos ny$$

$$w = h_0(x)\, \Sigma a_n{}^\circ \sin ny + h(x)\, \Sigma a_n \sin ny + \Sigma a_{mn} \sin mx \sin ny =$$
$$g_0(y)\, \Sigma b_m{}^\circ \sin mx + g(y)\, \Sigma b_m \sin mx + \Sigma b_{mn} \sin mx \sin ny$$

$$g(y) = h(y), \quad q(x) = h'(x), \quad r(y) = g'(y)$$

Here $h_0(x) = \pi/4$, $h(x) = x(\pi - x)/8$, if $P^3(x, y)$ is a function even in x in $[0, \pi]$, $h_0(x) = (\pi - 2x)/4$, $h(x) = x(\pi - x)(\pi - 2x)/24$, if $P^3(x, y)$ is a function odd in x in $[0, \pi]$. The choice of the polynomials is governed by the ease with which they can be expanded in Fourier series in the corresponding systems of functions.

The boundary conditions (2.3) yield $a_n{}^\circ \equiv 0$, $b_m{}^\circ \equiv 0$. According to the results of Sect.1, all series in the expressions for u and v can be differentiated term by term after their substitution into (2.1). The series in the first representation for w can be differentiated term by term up to the fourth order in x and second order in y. The series in the second representation for w can be differentiated term by term up to the fourth order in y and second order in x.

In order to determine ten groups of unknowns $\alpha_n, \alpha_{0n}, \alpha_{mn}, \delta_m, \delta_{m0}, \delta_{mn}, a_n, a_{mn}, b_m, b_{mn}$, we have five relations obtained after substituting (2.5) into system (2.1), four relations obtained after substituting (2.5) into the boundary conditions (2.4), and the relation which follows from the fact that two representations for w are identical. Let the indices i and j take, from now on, the values 1, 2, 3. We obtain the elements D_{mn}^{ij} of the matrix D_{mn} from the relations

$$D_{mn}^{i1} \cos mx \sin ny = L_{i1}(\cos mx \sin ny) \tag{2.6}$$

$$D_{mn}^{i2} \sin mx \cos ny = L_{i2}(\sin mx \cos ny)$$

$$D_{mn}^{i3} \sin mx \sin ny = L_{i3}(\sin mx \sin ny)$$

Let us denote by Δ_{mn}^{ij} the cofactors of the elements of the matrix D_{mn}^{ij} referred to the determinant of the matrix D_{mn}. Having written out the ten groups of relations mentioned above and transforming them in an appropriate manner, we obtain the solution of problem (2.1), (2.3) and (2.4) in the form

$$\begin{aligned}
&u(x,\ y) = \Sigma\alpha_{0n}\sin ny - \Sigma S^1_{mn}\cos mx\sin ny \\
&v(x,\ y) = \Sigma\delta_{m0}\sin mx - \Sigma S^2_{mn}\sin mx\cos ny \\
&w(x,\ y) = -\Sigma S^3_{mn}\sin mx\sin ny \\
&S^i_{mn} = \Delta^{1i}_{mn}\bar{\xi}_{mn} + \Delta^{2i}_{mn}\bar{\eta}_{mn} + \Delta^{3i}_{mn}\bar{\varkappa}_{mn} \\
&\bar{\xi}_{mn} = \xi_n + P^1_{mn},\ \bar{\eta}_{mn} = \eta_m + P^2_{mn},\ \bar{\varkappa}_{mn} = m\zeta_n + n\varkappa_m + P^3_{mn} \\
&\alpha_{0n} = \alpha^1_{0n}\xi_n + \alpha^2_{0n},\ \delta_{m0} = \delta^1_{m0}\eta_m + \delta^2_{m0}
\end{aligned} \tag{2.7}$$

The quantities $\alpha^k_{0n}, \delta^k_{m0}$ $((k = 1, 2)$ are expressed in terms of P^1_{0n}, P^2_{m0} and the constant coefficients of the system (2.1). The quantities $\zeta_n, \eta_m, \zeta_n, \varkappa_m$ represent the solution of the infinite algebraic system of equations (the indices m, n $\Delta^{ij}_{mn}, T^j_{mn}$) are omitted from the system)

$$\begin{aligned}
&\Big(\sum_m \Delta^{11} - \alpha^1_{0n}\Big)\xi_n + \sum_m m\Delta^{31}\zeta_n + \sum_m (\Delta^{21}\eta_m + \Delta^{31}\varkappa_m) = \alpha^2_{0n} - \sum_m T^1 \\
&\Big(\sum_n \Delta^{22} - \delta^1_{m0}\Big)\eta_m + \sum_n n\Delta^{32}\varkappa_m + \sum_n (\Delta^{12}\xi_n + \Delta^{32}\zeta_n) = \delta^2_{m0} - \sum_n T^2 \\
&\sum_m m\Delta^{13}\xi_n + \sum_m m\Delta^{33}\zeta_n + \sum_m m(\Delta^{23}\eta_m + \Delta^{33}\varkappa_m) = -\sum_m mT^3 \\
&\sum_n n\Delta^{23}\eta_m + \sum_n n\Delta^{33}\varkappa_m + \sum_n n(\Delta^{32}\xi_n + \Delta^{33}\zeta_n) = -\sum_n nT^3 \\
&T^j = \sum_{i=1}^{3}\Delta^{ij}P^i
\end{aligned} \tag{2.8}$$

The solution of the initial problem (2.1), (2.2) represents the sum of four solutions of problem (2.1), (2.3), (2.4) for the component loads P^1, P^2, P^3 of different, strictly defined evenness.

The infinite system (2.8) can be solved using the reduction method /4/. In a number of specific problems the system can be reduced to a regular form, and to justify the use of the reduction method it is sufficient that the order of decrease of the Fourier coefficients $P_{0n}{}^1$ but not less than $1/n$, P^2_{m0}, not less than $1/m$, $P^1_{mn}, P^2_{mn}, P^3_{mn}$, not less than $1/(mn)$/4/.

We note that if we write $D^{13}_{mn} = D^{23}_{mn} = D^{31}_{mn} = D^{32}_{mn} \equiv 0$, then the last two groups of equations of system (2.8) will yield an infinite system of equations obtained in /3/ and corresponding to the problem of the flexure of a rigidly clamped plate. The first two groups of equations of system (2.8) will, in this case, yield an infinite system of equations corresponding to the plane problem of the theory of elasticity.

The proposed method can be used to solve the problems of the theory of plates and shallow shells for various types of boundary conditions.

3. Numerical results.

Let us give the results of solving the problem of a shallow circular cylindrical shell rigidly clamped along the contour, in the region $\bar{\sigma} = [0, l_x] \times [0, l_y]$. The shell is described by a well-known system of equations /5/. Making the change of variable $x = \pi\bar{x}/l_x$, $y = \pi\bar{y}/l_y$ which transfer the region $\bar{\sigma}$ into σ we arrive at problem (2.1), (2.2). Let us write $\gamma = l_y/(R\pi)$, $\beta = l_y/l_x$ and set $h/R = 10^{-2}$, $\nu = 1/3$, $\beta = 1$. Here h is the shell thickness, R is its radius and ν is Poisson's ratio.

γ	$y;\ x = \pi/8$	$\pi/4$	$3\pi/8$	$\pi/2$
$1/(2\pi)$	0.374; 0.479	1.02; 1.20	1.52; 1.58	1.69
$6/(5\pi)$	0.590; 1.01	1.27; 1.27	1.40; 1.32	1.35
$1/(2\pi)$	0.311; 0.526	1.13; 1.41	1.96; 2.06	2.30
$6/(5\pi)$	−2.07; 2.02	−0.502; 4.41	4.21; 6.00	6.58

The table gives the values of dimensionless deflections w for various loads and various values of γ. The deflection sought is $\bar{w}(\bar{x}, \bar{y}) = (Eh)^{-1}(1 - \nu^2)R^2qw(x, y)$, E is Young's modulus, and q is a constant load. The loads $P^1 = P^2 = 0$. The upper half of the table corresponds to the load $P^3 = q$, and the lower half to $P^3 = q\sin x\sin y$. The first (second) number in each box gives the value of dimensionless deflection $w(\pi/2, y)$ $(w(x, \pi/2))$ for the corresponding values of y (of

x).

The system of Eqs.(2.8) is solved by the reduction method /4/. Here m and n in (2.8) and in the sums (2.7) take the values $1, 3, 5, \ldots, 19$. The first three signs of the appropriate solution do not change when the order of summation in (2.7) is increased as well as the corresponding number of equations in the reduced system (2.8).

We shall also give the value of the deflection $\bar{w}$ at the centre of the shell for $\beta = 10$, $\gamma = 1/(2\pi)$ and the load $P^3 = q$, retaining the previous values of the remaining parameters of the shell. The deflection at the centre $\bar{w}(l_x/2, l_y/2) = 3901 q l_x/E$ is identical with the known value /6/ of the deflection at the centre for the flexure of a beam of length l_x, width b and height h, under the load qb.

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Translated by L.K.